Journal of Approximation Theory **97**, 92–112 (1999)
Article ID jath.1997.3263, available online at http://www.idealibrary.com on IDEAL®

Vector Padé-Type Approximants and Vector Padé Approximants

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Communicated by Doron S. Lubinsky

Received May 15, 1997; accepted in revised form March 17, 1998

The aim of this paper is to define vector Padé-type approximants and vector Padé approximants following the same ideas as in the scalar case. This approach will be possible using Clifford's algebra structures. Vector Padé approximants will be derived from the theory of formal vector orthogonal polynomials. Connections between generalised inverse Padé approximants of Graves-Morris and vector-valued Padé approximants of Roberts will be given. New results will be proved.

1. INTRODUCTION

There are various definitions for vector Padé approximants. We are concerned with those suggested by Wynn [29, 30]. These vector Padé approximants are closely connected to the vector ε-algorithm. In fact, Wynn [29, 30] introduced continued fractions in a non-commutative algebra. He also defined vector continued fractions with the Samelson inverse. He was led to use McLeod isomorphism between vectors and some matrices (Clifford numbers) [18] for establishing results for the Padé-approximants. Then, this theory was developed by Graves-Morris in [10, 11] and Graves-Morris and Jenkins in [12]. Using the theory of vector continued fractions, Graves-Morris gave an axiomatic approach to vector-valued rational interpolants and results on Padé approximation. In particular, a five-term recurrence relationship was established for the denominator polynomials. All results about these vectors are generalizations of the corresponding ones for the scalar case. However, the algebraic structures cannot be generalized. For example, no three-term recurrence relation was obtained.

In [21, 22], Roberts gave another approach to vector Padé approximants, algebraically equivalent to the scalar case, using Clifford algebra.

0021-9045/99 $30.00

He provided a three-term recurrence relationship for the numerators and the denominators of diagonal and subdiagonal vector Padé approximants. He established that the two approaches are identical for real analytic data.

No approach using orthogonal polynomials was given for these vector Padé approximants.

The aim of this paper is to fill up this gap. A general framework will be given, similar to the scalar [4] and the non-commutative cases [7, 8]. Thus, using Clifford algebra, the construction of vector Padé-type approximants and vector Padé approximants will be presented. The approach is based on formal vector orthogonal polynomials [24] for deriving vector Padé approximants. Then, we shall establish that these vector Padé approximants coincide with those of Graves-Morris and Roberts [10, 21].

The formal vector orthogonal polynomials will allow us to give new results for the vector Padé approximants by exploiting the orthogonality. In particular, we shall give expressions for the numerator and the denominator of the vector Padé approximants in terms of designants, necessary and sufficient conditions of existence and uniqueness, new recurrence relationships, and others properties.

The vector Padé-type approximants can be interesting (as in the scalar case) for studying the poles of a vector-valued function, since, using Clifford algebra, the analogy with the scalar case could be exploited. However, this point will not be treated here.

2. CLIFFORD ALGEBRA AND GROUP, DESIGNANTS

Let us first recall some definitions and elementary properties of a Clifford algebra, group and designants.

2.1. *Clifford Algebra and Group*

Let $\{e_1, ..., e_d\}$ be an orthonormal basis of the Euclidean real vector space $\mathbb{R}^d$. The usual scalar product of two vectors $x, y \in \mathbb{R}^d$ will be denoted by (x, y). The universal real Clifford algebra associated to $\mathbb{R}^d$ is a unitary, associative but non-commutative (for $d > 1$) Algebra $\mathbf{A}_d$, containing $\mathbb{R}$ and $\mathbb{R}^d$ (see [1, 20]), for which, the Moore generalized inverse [19] of $x \in \mathbb{R}^d$ coincides with the inverse of x in $\mathbf{A}_d$ i.e,

$$\forall x \in \mathbb{R}^d, \qquad x^2 = (x, x). \tag{1}$$

$\mathbf{A}_d$ is a real vector space of dimension 2^d spanned by the products

$$e_{i_1} \cdots e_{i_r}, \qquad 0 \leqslant i_1 < \cdots < i_r \leqslant d.$$

There are various matrix representations of e_i (see, for example, [16, 18, 29]).

Setting $x = \sum_{i=1}^{d} x_i e_i$ and $y = \sum_{i=1}^{d} y_i e_i$, we have the main relations

$$xy + yx = 2(x, y) \tag{2}$$

and

$$xy = (x, y) + \sum_{i<j} (x_i y_j - x_j y_i)\, e_i e_j. \tag{3}$$

It is easy to see that $\mathbf{A}_d$ is not a division algebra; for example,

$$1 - e_1 \neq 0, \qquad 1 + e_1 \neq 0, \qquad \text{and} \qquad (1 - e_1)(1 + e_1) = 0.$$

Premultiplying the relation (2) by x, we deduce

$$\forall x, y, z \in \mathbb{R}^d, \qquad xyx = 2(x, y)\, x - \|x\|^2\, y, \tag{4}$$

and similarly we obtain

$$\forall x, y, z \in \mathbb{R}^d, \qquad xyz + zyx \in \mathbb{R}^d. \tag{5}$$

Thus, the product xyx belongs to $\mathbb{R}^d$ for all x, y belonging to $\mathbb{R}^d$.

Let us set

$$G_d = \left\{ \prod_{i=1}^{m} u_i,\, m \in \mathbb{N}^*,\, u_i \in \mathbb{R}^d \backslash \{0\} \right\}.$$

G_d forms a group for the multiplicative law [1]. This group is called the Clifford group. The spinor norm is a generalization to G_d of the 2-norm in $\mathbb{R}^d$ defined in the following way.

Let $\tilde{}$ be the anti-automorphism on $\mathbf{A}_d$ defined on the basic elements by

$$\tilde{1} = 1, \qquad \tilde{e_i} = e_i, \qquad i = 1, ..., d$$

and (6)

$$(\widetilde{c_{i_1} \cdots e_{i_r}}) = e_{i_r} \cdots e_{i_1}, \qquad \text{for} \quad 0 < i_1 < \cdots < i_r \leqslant d.$$

We obtain immediately

$$\forall u \in \mathbb{R}^d, \forall x \in \mathbf{A}_d, \forall y \in \mathbf{A}_d, \qquad \tilde{u} = u,\, \widetilde{xy} = \tilde{y}\tilde{x},\, \tilde{\tilde{x}} = x. \tag{7}$$

Let u be an element of G_d. Then there exists $u_1, ..., u_r \in \mathbb{R}^d \backslash \{0\}$ such that $u = u_1 ... u_r$. We have

$$u\tilde{u} = \|u\|^2 = u_1 \cdots u_r \widetilde{u_r} \cdots \widetilde{u_1} = \|u_1\|^2 \cdots \|u_r\|^2. \tag{8}$$

The spinor norm is the map $\|\cdot\|$ defined from G_d into $\mathbb{R}^+$ by $\|u\| = \sqrt{u\tilde{u}}$. In general, $u\tilde{u}$ does not belong to $\mathbb{R}^+$ when u is an arbitrary element of $\mathbf{A}_d$.

2.2. *Designants*

In the scalar case, Padé approximants have a determinantal representation (see [2, 4]), but Dyson [9] showed that there is no determinantal theory in a non-commutative algebra.

Thus, for providing formally a similar formula in our case, we need to use designants. They were introduced by Heyting [15]. They correspond to Gaussian elimination in a non-commutative algebra. We recall here only the definition for a system of two linear equations in two unknowns. For the general case and more details, see [15, 25].

Consider the system in $x_1, x_2 \in \mathbf{A}_d$, with coefficients on the right

$$\begin{cases} x_1 a_{11} + x_2 a_{12} = b_1, \\ x_1 a_{21} + x_2 a_{22} = b_2, \end{cases} \qquad a_{ij}, b_i \in \mathbf{A}_d, \qquad i, j = 1, 2. \tag{9}$$

Suppose that a_{11} is invertible, then by eliminating the unknown x_1 in the second equation of the system, we get

$$x_2(a_{22} - a_{12} a_{11}^{-1} a_{21}) = b_2 - b_1 a_{11}^{-1} a_{21}. \tag{10}$$

Set

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_r = a_{22} - a_{12} a_{11}^{-1} a_{21}. \tag{11}$$

It is an element of $\mathbf{A}_d$ which is called the right designant of the right system (9). If this designant is invertible then the system has a unique solution and x_2 is obviously given by the

$$x_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}_r \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_r^{-1}. \tag{12}$$

A similar construction exists for a left designant.

3. VECTOR PADÉ-TYPE APPROXIMANTS

Scalar Padé-type approximants and their link to general orthogonal polynomials are extensively studied by Brezinski and other authors [2, 4]. This study was generalized to the non-commutative case by Draux [7, 8]. There exist several generalizations to the vector case (see, for example, [13] for a review). Our aim here is to provide an orthogonal theory for the existing vector Padé approximants of Graves-Morris and Roberts. The approach is the following: the real vector space $\mathbb{R}^d$ is considered as a subset of the universal Clifford algebra A_d. Since A_d is non-commutative theoretical results on the subject will be applied. Then, the connection with the vector valued Padé approximants of Graves-Morris and Roberts will be established. We shall also obtain new results for these approximants.

Let us define Padé-type approximants as done by Brezinski [4] in the scalar case and Draux [8] in the non-commutative one.

Let $\mathbb{P}$ denote the set of polynomials in one real variable whose coefficients belong to $\mathbf{A}_d$ and let $\mathbb{P}_k$ denote the set of elements of $\mathbb{P}$ of degree less than or equal to k.

We consider the formal vector power series

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n + \cdots, \qquad c_i \in \mathbb{R}^d, \tag{13}$$

and a polynomial of $\mathbb{P}_k$ of degree k

$$v(x) = \sum_{i=0}^{k} b_i x^i. \tag{14}$$

The coefficient b_k is assumed to be invertible. In this case, v is said to be quasi-monic. We define the polynomial w by

$$w(t) = l\left(\frac{v(x) - v(t)}{x - t}\right), \tag{15}$$

where l is the left $\mathbf{A}_d$-linear functional, acting on x, defined as [24]

$$l\colon \mathbb{P} \to \mathbf{A}_d, \qquad \lambda x^i \to l(\lambda x^i) = c_i \lambda, \qquad \forall \lambda \in \mathbf{A}_d, \qquad \forall i \in \mathbb{N}. \tag{16}$$

Thus, w is a quasi-monic polynomial of $\mathbb{P}_{k-1}$ of degree $k-1$.

We set

$$\bar{v}(t) = t^k v(t^{-1}), \qquad \bar{w}(t) = t^{k-1} w(t^{k-1}). \tag{17}$$

Then, we obtain the

THEOREM 1. $\bar{w}(t)[\bar{v}(t)]^{-1} - f(t) = \mathcal{O}(t^k)$.

Proof. Expanding $(v(x)-v(t))/(x-t)$ and applying l, we get

$$w(t)=\sum_{l=0}^{k-1}\left(\sum_{i=0}^{k-l-1} c_i b_{l+i+1}\right) t^l.$$

We deduce that

$$\tilde{w}(t)=\sum_{l=0}^{k-1}\left(\sum_{i=0}^{k-l-1} c_i b_{l+i+1}\right) t^{k-1-l}.$$

Computing the product $f(t)\,\tilde{v}(t)$, we obtain

$$\begin{aligned} f(t)\,\tilde{v}(t) &= \left(\sum_{i=0}^{\infty} c_i t^i\right)\left(\sum_{j=0}^{k} b_{k-j} t^j\right) \\ &= \sum_{m=0}^{\infty}\left(\sum_{i=0}^{m} c_i b_{k-m+i}\right) t^m. \end{aligned}$$

Since $\tilde{w}(t)=\sum_{m=0}^{k-1}\left(\sum_{i=0}^{m} c_i b_{k-m+i}\right) t^m$, we get immediately

$$f(t)\,\tilde{v}(t)-\tilde{w}(t)=\mathcal{O}(t^k). \quad \blacksquare$$

DEFINITION 1. $\tilde{w}[\tilde{v}]^{-1}$ is called a left vector Padé-type approximant and it is denoted by $(k-1/k)^{(l)}$.

v is called the generating polynomial of $(k-1/k)^{(l)}$.

Remarks. (1) Obviously, we have similar definitions and results for the right linear functional r [24]. More precisely, if instead of l, we consider r and if we define w as

$$w(t)=r\left(\frac{v(x)-v(t)}{x-t}\right), \tag{18}$$

where r is the right $\mathbf{A}_d$-linear functional, acting on x, defined by

$$r\colon \mathbb{P}\to\mathbf{A}_d, \qquad \lambda x^i \to r(\lambda x^i)=\lambda c_i, \qquad \forall\lambda\in\mathbf{A}_d,\ \forall i\in\mathbb{N}, \tag{19}$$

then w is a quasi-monic polynomial of $\mathbb{P}_{k-1}$ of degree $k-1$.

The same proof as above leads to

$$[\tilde{v}(t)]^{-1}\,\tilde{w}(t)-f(t)=\mathcal{O}(t^k).$$

$[\tilde{v}(t)]^{-1}\,\tilde{w}(t)$ is called a right vector Padé-type approximant and it will be denoted by $(k-1/k)^{(r)}$.

(2) In general, left and right vector Padé-type approximants are different. Let us give a simple example: we take an arbitrary generating polynomial of degree 1

$$v(t) = b_0 + b_1 t, \qquad \bar{v}(t) = b_0 t + b_1.$$

For the left case, we have

$$w(t) = l\left(\frac{v(x) - x(t)}{x - t}\right) = l(b_1) = c_0 b_1, \qquad \bar{w}(t) = c_0 b_1,$$

and

$$(0/1)^{(l)} = c_0 b_1 [b_0 t + b_1]^{-1}.$$

For the right case, we have

$$w(t) = r\left(\frac{v(x) - x(t)}{x - t}\right) = r(b_1) = b_1 c_0, \qquad \bar{w}(t) = b_1 c_0,$$

and

$$(0/1)^{(r)} = [b_0 t + b_1]^{-1} b_1 c_0.$$

As the multiplicative law is non-commutative, $(0/1)^{(l)}$ and $(0/1)^{(r)}$ are different. We shall show, in the sequel, that this situation is totally different for vector Padé approximants: left vector Padé approximants coincide with right vector Padé approximants.

(3) In general, for an arbitrary choice of v, $(k-1/k)^{(l)}$ and $(k-1/k)^{(r)}$ do not belong necessarily to $\mathbb{R}^d$ for all t in $\mathbb{R}$. For example, we have

$$(0/1)^{(l)}(t) = c_0[1 + b_0 b_1^{-1} t]^{-1} = c_0 \sum_{i=0}^{\infty} (-1)^i (b_0 b_1^{-1})^i t^i.$$

The elements $(b_0 b_1^{-1})^i \in \mathbf{A}_d$ do not belong necessarily to $\mathbb{R}^d$. This is tedious, since we desire that the expansion of $(k-1/k)^{(l)}$ into a formal power series must be of the same nature than f. However, this situation can be always circumvented by choosing v with real coefficients.

(4) The remark (3) does not occur for vector Padé approximants as will be shown in the sequel. The expansion of a vector Padé approximant of f into a formal power series is of the same nature as f.

(5) The introduction of vector Padé-type approximants is motivated by the same considerations as in the scalar case (poles).

We shall now define left and right vector Padé-type approximants with a numerator and a denominator of arbitrary degrees.

By convention, we take $c_i = 0$ for $i < 0$ and $\sum_{i=0}^{m} s_i = 0$ for $m < 0$. Let us denote by f_m the formal power series

$$f_m(t) = \sum_{j=0}^{\infty} c_{m+j} t^j, \qquad \forall m \in \mathbb{Z}.$$

So, we have

$$t^m f_m(t) = \sum_{j=0}^{\infty} c_{m+j} t^{j+m} = f(t) - \sum_{j=0}^{m-1} c_j t^j.$$

Let $(k-1/k)_{f_m}^{(l)}$ (respectively $(k-1/k)_{f_m}^{(r)}$) be the left (respectively the right) vector Padé-type approximants of f_m.

Definition 2. $\sum_{i=0}^{m-1} c_i t^i + t^m (k-1/k)_{f_m}^{(l)}(t)$ (respectively $\sum_{i=0}^{m-1} c_i t^i + t^m (k-1/k)_{f_m}^{(r)}(t)$) is called the left (respectively the right) vector Padé-type approximant of type $(k-1+m/k)$ and denoted by $(k-1+m/k)_f^{(l)}$ (respectively $(k-1+m/k)_f^{(r)}$).

Let us denote by $l^{(i)}$ and $r^{(i)}$ the linear functionals defined by

$$\forall \lambda \in \mathbf{A}_d,\ \forall j \in \mathbb{N}, \qquad l^{(i)}(\lambda x^j) = c_{i+j}\lambda, \qquad r^{(i)}(\lambda x^j) = \lambda c_{i+j}. \tag{20}$$

We recall the construction of $(k-1/k)_{f_m}^{(l)}(t) = \tilde{w}(t)[\tilde{v}(t)]^{-1}$, but with the replacements of $f \to f_m$, $l \to l^{(m)}$ where

$$w(t) = l^{(m)}\left(\frac{v(x) - v(t)}{x - t}\right).$$

So,

$$\tilde{w}(t) = t^{k-1} l^{(m)}\left(\frac{v(x) - v(t^{-1})}{x - t^{-1}}\right),$$

and

$$(k-1/k)_{f_m}^{(l)}(t) = t^k l^{(m)}\left(\frac{v(t^{-1}) - v(x)}{1 - tx}\right)[\tilde{v}(t)]^{-1}.$$

Similar formulae exist for right vector Padé-type approximants of type $(k-1+m/k)$

$$(k-1/k)_{f_m}^{(r)}(t) = t^k[\bar{v}(t)]^{-1}\, r^{(m)}\left(\frac{v(t^{-1})-v(x)}{1-tx}\right).$$

We have the following result

THEOREM 2. *For all* $k \in \mathbb{N}$, $\forall m \in \mathbb{Z}$ *such that* $m \geqslant -k+1$,

$$(k-1+m/k)_f^{(l)}(t) - f(t) = \mathcal{O}(t^{k+m}), \tag{21}$$

$$(k-1+m/k)_f^{(r)}(t) - f(t) = \mathcal{O}(t^{k+m}). \tag{22}$$

Proof. From Theorem 1, we have $\forall k \in \mathbb{N}$, $\forall m \in \mathbb{Z}$, $(k-1/k)_{f_m}^{(l)}(t) - f_m(t) = \mathcal{O}(t^k)$. Since $k+m>0$, we get $t^m(k-1/k)_{f_m}^{(l)}(t) - t^m f_m(t) = \mathcal{O}(t^{k+m})$ from which we obtain $(k-1+m/k)_f^{(l)}(t) - f(t) = \mathcal{O}(t^{k+m})$. The same proof is valid for (22). ∎

We will now give some properties of left vector Padé-type approximants. The superscript (l) in the notation will be omitted. Similar results hold for the right case. In the sequel, we shall only treat the left case.

Let us denote by

$$h(t) = \sum_{i=0}^{\infty} d_i t^i$$

the reciprocal power series of f, that is, the series satisfying

$$h(t)\, f(t) = f(t)\, h(t) = 1. \tag{23}$$

From (1), we have

$$h(t) = \frac{f(t)}{\|f(t)\|^2}. \tag{24}$$

Thus h exists if and only if $\|c_0\| \neq 0$ and, in that case, the coefficients d_i belong to $\mathbb{R}^d$. The coefficients d_i are given by solving the linear system in $\mathbf{A}_d$

$$\begin{cases} d_0 c_0 = 1 \\ \sum_{j=0}^{i} d_j c_{i-j} = 0, \qquad i \geqslant 1. \end{cases} \tag{25}$$

PROPERTY 1. *Let* $V(t) = c_0 v(t) + w^{(1)}(t)$ *be the generating polynomial of* $(k/k)_h$, *with* $w^{(1)}(t) = l^{(1)}((v(x) - v(t))/(x - t))$. *We have*

$$(k/k)_h(t)(k/k)_f(t) = (k/k)_f(t)(k/k)_h(t) = 1.$$

Proof. We have

$$(k/k)_f(t) = \bar{V}(t)[\bar{v}(t)]^{-1} = \sum_{i=0}^{k} c_i t^i + \mathcal{O}(t^{k+1}).$$

Let $\bar{q}(t)$ be the numerator polynomial for $(k/k)_h$. Then $(k/k)_h$ can be written as

$$(k/k)_h(t) = \bar{q}(t)[\bar{V}(t)]^{-1} = \sum_{i=0}^{k} d_i t^i + \mathcal{O}(t^{k+1}).$$

Thus

$$(k/k)_h(t)(k/k)_f(t) = 1 + \mathcal{O}(t^{k+1}),$$

and

$$\bar{q}(t)[\bar{v}(t)]^{-1} = 1 + \mathcal{O}(t^{k+1}).$$

We obtain

$$\bar{q}(t) = \bar{v}(t) + \mathcal{O}(t^{k+1}).$$

As $\bar{q}$ is a polynomial of degree k, we obtain $\bar{q} = \bar{v}$. ∎

PROPERTY 2. *Let* W *be a polynomial of degree* p *and* V *a polynomial of degree* q *such that* $V(0)$ *is invertible and* $W(t)[V(t)]^{-1} - f(t) = \mathcal{O}(t^{p+1})$. *Then* $W(t)[V(t)]^{-1} = (p/q)_f(t)$ *with the generating polynomial* $\bar{V}$.

Proof. We have

$$W(t)\, V(t) - f(t) = \mathcal{O}(t^{p+1}),$$

and

$$(p/q)_f(t) = \sum_{i=0}^{p-q} c_i t^i + t^{p-q+1}(q-1/q)_{f_{p-q+1}}(t).$$

Then $(p/q)_f$ can be written

$$(p/q)_f(t) = \bar{r}[\bar{\bar{V}}(t)]^{-1} = \bar{r}[V(t)]^{-1},$$

with the degree of r equal to p.

Since

$$W(t)[V(t)]^{-1} = \bar{r}(t)[V(t)]^{-1} + \mathcal{O}(t^{p+1}),$$

we obtain

$$W(t) = \bar{r} + \mathcal{O}(t^{p+1}).$$

As r is a polynomial of degree p it follows $\bar{r} = W$. ∎

We can also give properties of linearity and homographic covariance and a compact formula deduced from that of Nuttall. Let us now give a formula for the error.

THEOREM 3. *The error of left vector Padé-type approximation is given by*

$$\begin{aligned} f(t) - (k-1/k)_f(t) &= t^k l(v(x)(1-xt)^{-1})(\tilde{v}(t))^{-1} \\ &= t^k \left(\sum_{i=0}^{\infty} \mu_i t^i \right) (\tilde{v}(t))^{-1}, \end{aligned}$$

with $\mu_i = l(x^i v(x))$.

Proof. From $w(t) = l((v(x) - v(t))/(x-t))$, it is easy to see that

$$\begin{aligned} \tilde{w}(t) &= t^k l\left(\frac{v(t^{-1})}{1-tx}\right) - t^k l\left(\frac{v(x)}{1-tx}\right) \\ &= l\left(\frac{\tilde{v}(t)}{1-tx}\right) - t^k l\left(\frac{v(x)}{1-tx}\right) \\ &= f(t)\,\tilde{v}(t) - t^k l\left(\frac{v(x)}{1-tx}\right). \end{aligned}$$

The first equality of the theorem is obtained. Then, expanding $v(x)/(1-tx)$ into a power series and applying the functional l, we get

$$\begin{aligned} \tilde{w}(t) &= f(t)\,\tilde{v}(t) - t^k \left(\sum_{i=0}^{\infty} l(v(x)\,x^i)\,t^i \right) \\ &= f(t)\,\tilde{v}(t) - t^k \left(\sum_{i=0}^{\infty} \mu_i t^i \right). \end{aligned}$$

Thus, we obtain the second equality of the theorem

$$\bar{w}(t)(\bar{v}(t))^{-1} = f(t) - t^k \left(\sum_{i=0}^{\infty} \mu_i t^i \right) [\bar{v}(t)]^{-1}. \quad \blacksquare$$

More generally, we have the

THEOREM 4. *Let v denote the generating polynomial of $(p/q)_f$. We have the error formula*

$$f(t) - (p/q)_f(t) = t^{p+1} l^{(p-q+1)}(v(x)(1-tx)^{-1})[\bar{v}(t)]^{-1}.$$

The proof is the same as above.

4. VECTOR PADÉ APPROXIMANTS

For obtaining a higher order of approximation, we see from the error formula that we can choose the generating polynomial v such that

$$l(x^i v(x)) = 0, \qquad \text{for} \quad i = 0, ..., m \leqslant k-1. \tag{26}$$

Since v has the degree k, m cannot be greater than $k-1$. For the maximum value $m = k-1$, we obtain the left vector Padé approximant which will be denoted by $[k-1/k]_f^{(l)}$. If instead of l, we consider r, we obtain the right vector Padé approximant, which will be, denoted by $[k-1/k]_f^{(r)}$. The generating polynomial v satisfies

$$l(x^i v(x)) = 0, \qquad \text{for} \quad i = 0, ..., k-1. \tag{27}$$

The relation (26) can be written

$$\sum_{j=0}^{k} c_{i+j} b_j = 0, \qquad i = 0, ..., k-1. \tag{28}$$

Thus, as in the scalar case, the polynomial v is orthogonal with respect to the left linear functional l.

Let us denote by $L_k^{(n)}$ (respectively by $R_k^{(n)}$) the orthogonal polynomial of degree k with respect to the left linear functional $l^{(n)}$ (respectively to the right linear functional $r^{(n)}$) and $K_k^{(n)}$ (respectively $Q_k^{(n)}$) its associated polynomial.

We set

$$M_k^{(n)} = \begin{pmatrix} c_n & \cdots & c_{n+k-1} \\ \vdots & & \vdots \\ c_{n+k+1} & \cdots & c_{n+2k-2} \end{pmatrix}.$$

$M_k^{(n)}$ is the matrix of the system connected to the orthogonality system [24]

$$l^{(n)}(x^i L_k^{(n)}) = 0, \qquad \text{for} \quad i = 0, ..., k-1. \tag{29}$$

$l^{(n)}$ (respectively $r^{(n)}$) is said to be definite if the matrix $M_k^{(n)}$ is invertible (see [24] for other characterizations and details).

The error formula for the vector Padé approximant becomes

THEOREM 5. $f(t) - [k-1/k]_f(t) = t^{(2k)} l(x^k L_k(x)(1-xt)^{-1})[\bar{L}_k(t)]^{-1}$.

The proof is obvious from Theorem 4.

Theorem 5 gives us the

COROLLARY 1. *If l is definite, then the left vector Padé approximant $[k-1/k]_f^{(l)}$ is equal to the right vector Padé approximant $[k-1/k]_f^{(r)}$.*

Proof. From Theorem 4, we have

$$f(t) - [k-1/k]_f^{(l)} = \mathcal{O}(t^{2k}) \qquad \text{and} \qquad f(t) - [k-1/k]_f^{(r)} = \mathcal{O}(t^{2k}).$$

By subtracting the second equality from the first one, we obtain

$$\bar{K}_k(t)[\bar{L}_k(t)]^{-1} - [\bar{R}_k(t)]^{-1}\, \bar{Q}_k(t) = \mathcal{O}(t^{2k})$$

and thus, we have

$$\bar{R}_k(t)\, \bar{K}_k(t) - \bar{Q}_k(t)\, \bar{L}_k(t) = \mathcal{O}(t^{2k}).$$

As the left hand-side is a polynomial of degree less than $2k-1$, it is identically zero. ∎

As in the scalar case [4], we can give the

DEFINITION 3. The quantity

$$\sum_{i=0}^{p-q} c_i t^i + t^{p-q+1} \bar{K}_q^{(p-q+1)}(t)[\bar{L}_q^{(p-q+1)}(t)]^{-1}$$

is called the vector Padé approximant of type (p/q) and it will be denoted by $[p/q]_f$.

Thus, we obtain the error formula

THEOREM 6. $f(t) - [p/q]_f(t) = t^{p+q+1} l^{(p-q+1)}(x^i L_q^{(p-q+1)}(1-xt)^{-1}) [\bar{L}_q^{(p-q+1)}(t)]^{-1}$.

Proof. This is obvious from Theorem 4. ∎

All the properties of Section 3 are still valid for the vector Padé approximants, in particular we have

THEOREM 7. *If $M_k^{(n+1)}$ is invertible with $k \in \mathbb{N}$ and $n \in \mathbb{Z}$ such that $n \geqslant k-1$ and if h is the reciprocal series of f formally defined by $f(t)\, h(t) = 1$, then*

$$[k+n/k]_f(t)[k/n+k]_h(t) = [k/n+k]_h(t)[k+n/k]_f(t).$$

In [10], Graves-Morris provided a practical Thiele-fraction method for rational interpolation of vectors, based on the Samelson inverse. He showed that these rational interpolants can be computed recursively by the Claessens' ε-algorithm, implemented with the Samelson inverses. Vector-valued Padé approximants are showed as a limit case of these interpolants and can be computed by the standard vector ε-algorithm of Wynn.

In [21], Roberts gave another approach similar to the scalar case. It consisted in finding two polynomials p_m and q_n of degree respectively equal to m and n, whose coefficients are in $\mathbf{A}_d$, and such that $p_m(x)[q_n(x)]^{-1} - f(x) = \mathcal{O}(x^{n+m+1})$, where f is the power series given by (13). He adopted the Baker convention $q_n(0) = 1$. The construction of these approximants is based on vector continued fractions. He gave also a three-recurrence relationship for the numerator and the denominator of these approximants. However, this three-term recurrence relationship involved some vector α_k which is unknown. He established that these approximants coincide with those of Graves-Morris for real analytical data. No conditions of existence and no error formula were given.

Comparing with these vector Padé approximants, we see that they are identical to the Padé approximants defined above (by uniqueness). Since Baker convention [2] holds, we have equality between the numerators and the denominators of these two approximants.

As in the scalar case, the theory of orthogonality developed here, gives us a new insight into these approximants. It allowed us to give a new approach, a new formula of the error, conditions of the existence, and links with the theory of orthogonal polynomials in the non-commutative case. In the sequel, we shall easily obtain from this theory necessary and sufficient conditions of the normality, new recurrence relations for computing these approximants, and a *qd*-algorithm. Expressions with continued fractions are also given with explicit coefficients. Thus, we see that scalar theory is perfectly generalized to the vector case. Clifford algebra and orthogonality play a fundamental role.

5. NORMAL CASE

We shall use the following notations

$$[n+k/k]_f(t) = \bar{N}_k^{(n+1)}[\bar{L}_k^{(n+1)}(t)]^{-1},$$

where

$$\bar{N}_k^{(n+1)}(t) = \left(\sum_{i=0}^{n} c_i t^i\right) \bar{L}_k^{(n+1)}(t) + t^{n+1}\bar{K}_k^{(n+1)}(t). \tag{30}$$

Let us display the vector Padé approximants in a table as

$$\begin{array}{cccc} [0/0] & [0/1] & [0/2] & \cdots \\ [1/0] & [1/1] & [1/2] & \cdots \\ [2/0] & [2/1] & [2/2] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

DEFINITION 4. If $\forall k \in \mathbb{N}$, $\forall n \in \mathbb{Z}$, with $n \geqslant -k+1$, $M_k^{(n)}$ is invertible, the vector Padé table is said to be normal.

We have from [24]

THEOREM 8. *The four following properties are equivalent*:

(i) $l^{(n)}[x^k L_k^{(n)}]$ *is invertible* $\forall k \in \mathbb{N}$, $\forall n \in \mathbb{Z}$, *with* $n \geqslant -k+1$.

(ii) $L_k^{(n)}(0)$ *is invertible*, $\forall k \in \mathbb{N}$, $\forall n \in \mathbb{Z}$, *with* $n \geqslant -k+1$ *and* c_0 *is invertible.*

(iii) $\bar{N}_k^{(n)}(0)$ *is invertible*, $\forall k \in \mathbb{N}$, $\forall n \in \mathbb{Z}$, *with* $n \geqslant -k+1$ *and* c_0 *is invertible.*

(iv) *The vector Padé table is normal.*

This theorem presents a drawback: in general, it is not easy to know if an element of $\mathbf{A}_d$ is invertible or not.

Let us state a theorem giving a sufficient condition for the invertibility required in Theorem 8. We shall denote by h_k^n $\forall k \in \mathbb{N}$, $\forall n \in \mathbb{Z}$ with $n+k \geqslant 1$, the right Hankel designant

$$\begin{vmatrix} c_n & \cdots & c_{n+k} \\ \vdots & & \vdots \\ c_{n+k} & \cdots & c_{n+2k} \end{vmatrix}_r .$$

It was shown in [26] that this right Hankel designant h_k^n belongs to $\mathbb{R}^d$.

THEOREM 9. *For all* $k \in \mathbb{N}$, $\forall n \in \mathbb{Z}$ *with* $n+k \geqslant 1$, $l^{(n)}[x^k L_k^{(n)}]$ *is invertible if and only if* $\forall k \in \mathbb{N}$, $\forall n \in \mathbb{Z}$ *with* $n+k \geqslant 1$, $h_k^n \neq 0$.

Proof. See [24]. ∎

6. NEW ALGORITHMS, CONTINUED FRACTIONS, *qd*-ALGORITHM

In [24], we gave some recurrence relations between orthogonal polynomials. These relations can be used to derive new algorithms for computing recursively any sequence of vector Padé approximants in the table. From the relations (see [24])

$$\begin{aligned} xL_k^{n+1}(x) &= L_{k+1}^n(x) + L_k^n(x)\, q_{k+1}^n, \\ L_k^{n+1}(x) &= L_k^n(x) - L_{k-1}^{n+1}(x)\, e_k^n, \\ K_k^{n+1}(t) &= K_{k+1}^n(t) + K_k^n(t)\, q_{k+1}^n - c_n L_k^{n+1}(t), \\ K_k^{n+1}(t) &= tK_k^n(t) - c_n L_k^n(t) - K_{k-1}^{n+1}(t)\, e_k^n \end{aligned}$$

and since

$$\bar{L}_k^n(x) = x^k L_k^n(x^{-1}), \qquad \bar{K}_k^n(x) = x^{k-1} K_k^n(x^{-1}),$$

we derive

$$\bar{L}_{k+1}^n(x) = \bar{L}_k^{n+1}(x) - x\bar{L}_k^n(x)\, q_{k+1}^n, \tag{31}$$

$$\bar{L}_k^{n+1}(x) = \bar{L}_k^n(x) - x\bar{L}_{k-1}^{n+1}(x)\, e_k^n, \tag{32}$$

$$\bar{K}_{k+1}^n(x) = x\bar{K}_k^{n+1}(x) - x\bar{K}_k^n(x)\, q_{k+1}^n + c_n \bar{L}_k^{n+1}(x), \tag{33}$$

$$x\bar{K}_k^{n+1}(x) = \bar{K}_k^n(x) - x^2\bar{K}_{k-1}^{n+1}(x)\, e_k^n - c_n \bar{L}_k^n(x). \tag{34}$$

We begin first by the relations corresponding to a method due to Watson in the scalar case [28] to compute recursively approximants located on a descending staircase of the Padé table. We shall make use of the same notations as in [4, 8]: knowing the two Padé approximants denoted by $\bullet$, then the Padé approximant denoted by $*$ can be computed.

THEOREM 10. *We have*

$$\bar{N}_{k+1}^{(n)}(t)[\bar{L}_{k+1}^{(n)}(t)]^{-1} = \quad \begin{array}{cc} \bullet & \\ \bullet & * \end{array}$$

$$[\bar{N}_k^{(n+1)}(t) - t\bar{N}_k^{(n)}(t)\, q_{k+1}^n][\bar{L}_k^{(n+1)}(t) - t\bar{L}_k^{(n)}(t)\, q_{k+1}^n]^{-1} \tag{35}$$

$$\bar{N}_k^{(n+1)}(t)[\bar{L}_k^{(n+1)}(t)]^{-1} = \quad \begin{array}{cc} \bullet & \bullet \\ & * \end{array}$$

$$[\bar{N}_k^{(n)}(t) - t\bar{N}_{k-1}^{(n+1)}(t)\, e_k^n][\bar{L}_k^{(n)}(t) - t\bar{L}_{k-1}^{(n+1)}(t)\, e_k^n]^{-1}. \tag{36}$$

Proof. Given $\bar{K}_k$ in terms of $\bar{N}_k$ from (30) and using (31), we obtain

$$\bar{N}_{k+1}^n = \bar{N}_k^{n+1} - t\bar{N}_k^n q_{k+1}^n.$$

Thus (35) is immediate.

In the same way, we express $\bar{K}_k$ in terms of $\bar{N}_k$ by using (30), and by replacing in (34), we obtain

$$\bar{N}_k^{n+1} = \bar{N}_k^n - t\bar{N}_{k-1}^{n+1} e_k^n.$$

Thus, (36) is immediate by using (32). ∎

It is not our purpose here to give all the possible relations. Let us only say that the other relations exist for computing the vector Padé table in all directions exactly as in the scalar case. This point will be treated in a forthcoming paper. It is well known that the theory of orthogonal polynomials and the theory of continued fractions are closely connected in the scalar case since both satisfy a three-term recurrence relation. We shall show that this aspect is still true in the vector case.

Let us consider the monic vector orthogonal polynomials L_k and their associated polynomials K_k. From the three-term recurrence relationship satisfied by L_k and K_k [24], we obtain

$$\bar{L}_{-1}(x) = 0, \qquad \bar{L}_0(x) = 1,$$
$$\bar{K}_0(x) = 0, \qquad \bar{K}_1(x) = c_0,$$

$$\bar{L}_{k+1}(x) = \bar{L}_k(x)(1 + xB_{k+1}) + x^2\bar{L}_{K-1}(x)\, C_{k+1}; \qquad k \geqslant 0, \tag{37}$$

$$\bar{K}_{k+1}(x) = \bar{K}_k(x)(1 + xB_{k+1}) + x^2\bar{K}_{k-1}(x)\, C_{k+1}; \qquad k \geqslant 1. \tag{38}$$

Let us use the convention $a/b = ab^{-1}$ where $a, b \in \mathbf{A}_d$.

It is easy to see that, in the non-commutative case, the successive convergents

$$\frac{A_n}{B_n} = A_n[B_n]^{-1} = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n}}}$$

can be computed recursively by the relations

$$A_{n+1} = A_n b_{n+1} + A_{n-1} a_{n+1}, \qquad B_{n+1} = B_n b_{n+1} + B_{n-1} a_{n+1} \tag{39}$$

with

$$A_0 = 0, \qquad A_1 = a_1,$$
$$B_0 = 1, \qquad B_1 = b_1.$$

Let us consider the continued fraction

$$\bar{C}(x) = \frac{C_1}{1 + B_1 x +} \frac{C_2 x^2}{1 + B_2 x +} \frac{C_3 x^2}{1 + B_3 x +} \cdots,$$

and denote by

$$\bar{C}_k(x) = \frac{C_1}{1 + B_1 x +} \frac{C_2 x^2}{1 + B_2 x +} \frac{C_3 x^2}{1 + B_3 x +} \cdots \frac{C_k x^2}{1 + B_k x}$$

its successive convergents.

THEOREM 11. $\bar{C}_k(x) = [k-1/k]_f(x)$.

Proof. The assertion is true for $k = 1$.

Suppose it is true for k. Then, from the recurrence relations (39) among the successive convergents of a non-commutative continued fraction, we see immediately that

$$\bar{C}_{k+1}(x) = \frac{\bar{K}_k(x)(1 + B_{k+1} x) + \bar{K}_{k-1}(x)\, C_{k+1} x^2}{\bar{L}_k(x)(1 + B_{k+1} x) + \bar{L}_{k-1}(x)\, C_{k+1} x^2} = \frac{\bar{K}_{k+1}(x)}{\bar{L}_{k+1}(x)}. \quad \blacksquare$$

The continued fraction $\bar{C}$ is called the left vector continued fraction associated with the series f. This property can be extended to the other diagonals of the vector Padé table. Obviously, using the right vector orthogonal polynomials R_k and their associated polynomials Q_k, similar results exist with the convention $a/b = b^{-1}a$.

Let us now consider the continued fraction

$$\bar{D}(x) = \frac{c_0}{1-}\,\frac{q_1^0 x}{1-}\,\frac{e_1^0 x}{1-}\,\frac{q_2^0 x}{1-}\,\frac{e_2^0 x}{1-}\cdots$$

Using the algorithm given by (39), it is easy to see that the convergents $\bar{D}_k(x) = \bar{V}_k(x)/\bar{U}_k(x)$ of $\bar{D}$ satisfy

$$\bar{U}_{2k+1}(x) = \bar{U}_{2k}(x) - x\bar{U}_{2k-1}(x)\, e_k^0,$$
$$\bar{U}_{2k+2}(x) = \bar{U}_{2k+1}(x) - x\bar{U}_{2k}(x)\, q_{k+1}^0,$$

with $\bar{U}_0(x) = 1$ and $\bar{U}_1(x) = 1$. Similar relations hold for $\bar{V}_k$ with the initializations $\bar{V}_0(x) = 0$ and $\bar{V}_1(x) = c_0$. Thus comparing with the relations (31), (32), (33), and (34), we get the

THEOREM 12.

$$\bar{V}_{2k} = \bar{K}_k, \qquad \bar{V}_{2k+1} = c_0 \bar{L}_k^{(1)} + x\bar{K}_k^{(1)},$$
$$\bar{U}_{2k} = \bar{L}_k, \qquad \bar{U}_{2k+1} = \bar{L}_k^{(1)},$$

and

$$\bar{D}_{2k}(x) = [k-1/k]_f(x)$$
$$\bar{D}_{2k+1}(x) = [k/k]_f(x).$$

The continued fraction $\bar{D}$ is called the left vector continued fraction corresponding to the series f. The other descending staircases of the vector Padé table can be related to the corresponding left vector continued fractions in a similar way. For that purpose, we have to consider the continued fractions

$$\bar{D}^{(n)}(x) = c_0 + c_1 x + \cdots + c_n x^n + \frac{c_{n+1}}{1-}\,\frac{q_1^{n+1} x}{1-}\,\frac{e_1^{n+1} x}{1-}\,\frac{q_2^{n+1} x}{1-}\,\frac{e_2^{n+1} x}{1-}\cdots,$$

for $n \geqslant -1$. Designantal formulae for the e_k^n's and q_k^n's and some of their other properties are given in [24].

THEOREM 13. *We have*

$$\forall k \in \mathbb{N}, \quad \forall n \in \mathbb{Z} \quad \text{with} \quad n+k \geqslant 1, \quad \bar{K}_k^n \tilde{\bar{K}}_k^n \in \mathbb{R}[x], \tag{40}$$
$$\forall k \in \mathbb{N}, \quad \forall n \in \mathbb{Z} \quad \text{with} \quad n+k \geqslant 1, \quad \bar{L}_k^n \tilde{\bar{L}}_k^n \in \mathbb{R}[x], \tag{41}$$
$$\forall k \in \mathbb{N}, \quad \forall n \in \mathbb{Z} \quad \text{with} \quad n+k \geqslant 1, \quad \bar{K}_k^n \tilde{\bar{L}}_k^n \in \mathbb{R}^d[x], \tag{42}$$

where $\mathbb{R}[x]$ *(respectively* $\mathbb{R}^d[x]$*) denotes the set of polynomials with real coefficients (respectively vector coefficients) in one variable.*

Proof. It is shown in [14, 21] that

$$\forall k \in \mathbb{N},\ \forall n \in \mathbb{Z} \textit{ with } n+k \geqslant 1,$$
$$\bar{N}_k^n \tilde{\bar{N}}_k^n \in \mathbb{R}[x],\ \bar{L}_k^n \tilde{\bar{L}}_k^n \in \mathbb{R}[x],\ \bar{N}_k^n \tilde{\bar{L}}_k^n \in \mathbb{R}^d[x]. \tag{43}$$

From (30) we get $x^{2n}\bar{K}_k^n \tilde{\bar{K}}_k^n = (\bar{N}_k^n - (\sum_{i=0}^{n-1} c_i x^i)\, \bar{L}_k^n)(\tilde{\bar{N}}_k^n - \tilde{\bar{L}}_k^n(\sum_{i=0}^{n-1} c_i x^i))$.

Using (1), (2), and (43), we deduce $\bar{K}_k^n \tilde{\bar{K}}_k^n \in \mathbb{R}[x]$.

Since $x^n \bar{K}_k^n \tilde{\bar{L}}_k^n = (\bar{N}_k^n - (\sum_{i=0}^{n-1} c_i x^i)\, \bar{L}_k^n)\, \tilde{\bar{L}}_k^n$, we obtain from (43) that $\bar{K}_k^n \tilde{\bar{L}}_k^n \in \mathbb{R}^d[x]$. ∎

Thus, the vector orthogonal polynomials L_k^n and their associated polynomials K_k^n satisfy the same properties.

ACKNOWLEDGMENT

I am grateful to the referee for his constructive comments and suggestions.

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